On endomorphism algebras of functors with non-compact domain

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Abstract

As a development of [2] and [3], we construct a "VN-core" in $\mathbf{Vect}_k$ for each k-linear split-semigroupal functor from a suitable monoidal category $\mathcal{C}$ to $\mathbf{Vect}_k$. The main aim here is to avoid the customary compactness assumption on the set of generators of the domain category $\mathcal{C}$ (cf. [3]).

1 Introduction

We propose the construction of a VN-core associated to each (k-linear) split semigroupal functor U from a suitable monoidal category $\mathcal{C}$ to $\mathbf{Vect}_k$, where all our categories, functors, and natural transformations are assumed to be k-linear, for a fixed field k. Essentially, the category $\mathcal{C}$ must be equipped with a small "U-generator" $\mathcal{A}$ carrying some extra duality information and with UA still being finite dimensional for all A in $\mathcal{A}$.

We shall use the term "VN-core" (in $\mathbf{Vect}_k$) to mean a (usual) k-semibialgebra E together with a k-linear endomorphism S such that

$$\mu(\mu \otimes 1)(1 \otimes S \otimes 1)(1 \otimes \delta)\delta = 1 : E \to E.$$

The VN-core is called "antipodal" if $S(xy) = SySx$ (and $S(1) = 1$) for all $x, y \in E$. This minimal type of structure is introduced here in order to avoid compactness assumptions on the generator $\mathcal{A} \subset \mathcal{C}$ and, at the same time, retain the "fusion" operator, namely

$$(\mu \otimes 1)(1 \otimes \delta) : E \otimes E \to E \otimes E,$$

satisfying the usual fusion equation [7]. Note that here the fusion operator always has a partial inverse (see [1]).

In §2 we establish sufficient conditions on a functor U in order that

$$\mathrm{End}^{\vee} U = \int^{A} (UA)^* \otimes UA$$

be a VN-core in $\mathbf{Vect}_k$ (following [2]). This core can be completed to a VN-core $\mathrm{End}^{\vee} U \oplus k$ with a unit element. In §3 we give several examples of suitable functors U for the theory.

2 The construction of $\mathrm{End}^\vee U$

Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category and let

$$U : \mathcal{C} \to \mathbf{Vect}$$

be a functor with both a semigroupal structure, denoted

$$r = r_{C,D} : UC \otimes UD \to U(C \otimes D),$$

and a cosemigroupal structure, denoted

$$i = i_{C,D} : U(C \otimes D) \to UC \otimes UD,$$

such that $ri = 1$.

We shall suppose also that there exists a small full subcategory $\mathcal{A}$ of $\mathcal{C}$ with the properties:

1. UA is finite dimensional for all $A \in \mathcal{A}$,
2. U-density; the canonical map
$$\alpha_C : \int^A \mathcal{C}(A, C) \otimes UA \to UC$$
is an isomorphism for all $C \in \mathcal{C}$,
3. there is an "antipode" functor
$$(-)^* : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$$
with a ("canonical") map $e_A : A \otimes A^* \otimes A \to A$ in $\mathcal{C}$ for each $A \in \mathcal{A}$,
4. there is a natural isomorphism
$$u = u_A : U(A^*) \xrightarrow{\cong} U(A)^*,$$
5. the following diagrams defining $\tilde{\tau}$, $\tilde{\rho}$ both commute
$$\begin{tikzcd}
& UA \otimes U(A^*) \otimes UA \arrow[dr, "r_3"] & \\
UA \otimes U(A)^* \otimes UA \arrow[ur, "1\otimes u^{-1}\otimes 1"] \arrow[rr, dotted, "\tilde{\tau}"] \arrow[dr, "e_{UA}"'] & & U(A \otimes A^* \otimes A) \arrow[dl, "Ue_A"] \\
& UA &
\end{tikzcd}$$
and
$$\begin{tikzcd}
& UA \otimes U(A^*) \otimes UA \arrow[dl, "1\otimes u\otimes 1"'] & \\
UA \otimes U(A)^* \otimes UA \arrow[dr, "e_{UA}"'] & & U(A \otimes A^* \otimes A) \arrow[ul, "i_3"'] \arrow[ll, dotted, "\tilde{\rho}"'] \arrow[dl, "Ue_A"] \\
& UA &
\end{tikzcd}$$
where $e_{UA} = 1 \otimes \mathrm{ev}$ in $\mathbf{Vect}$, and $r_3 i_3 = 1$.

We now define the semibialgebra structure $(\mathrm{End}^\vee U, \mu, \delta)$ on

$$\mathrm{End}^\vee U = \int^A U(A)^* \otimes UA$$

as in [2] §2, with the isomorphism of k-linear spaces

$$S = \sigma : \mathrm{End}^\vee U \to \mathrm{End}^\vee U$$

given (as in [2] §3) by the usual components

$$\begin{tikzcd}
U(A)^* \otimes UA \arrow[r, "\sigma_A"] \arrow[d, "1\otimes d"'] & U(A^*)^* \otimes U(A^*) \\
U(A)^* \otimes U(A)^{**} \arrow[r, "u^{-1}\otimes u^*"] & U(A^*) \otimes U(A^*)^* \arrow[u, "c"']
\end{tikzcd}$$

where d is the canonical map from a vector space to its double dual. Furthermore, each map

$$e_{UA} = 1 \otimes \mathrm{ev} : UA \otimes UA^* \otimes UA \to UA$$

satisfies both the conditions

$$\begin{tikzcd}
 & UA \otimes UA^* \otimes UA \arrow[dr, "e_{UA}"] & \\
UA \arrow[ur, "n\otimes 1"] \arrow[rr, "1"'] & & UA
\end{tikzcd} \tag{E1}$$

commutes, and

$$\begin{tikzcd}
 & UA^* \otimes UA \otimes UA^* \arrow[dr, "1\otimes d\otimes 1"] & \\
UA^* \arrow[ur, "1\otimes n"] \arrow[rr, "e^*_{UA}"'] & & UA^* \otimes UA^{**} \otimes UA^*
\end{tikzcd} \tag{E2}$$

commutes, where $n = \mathrm{coev} : 1 \to UA \otimes UA^*$ in **Vect**.

Then we obtain:

Theorem 2.1. *The structure* $(\mathrm{End}^\vee U, \mu, \delta, S)$ *is a VN-core in* $\mathbf{Vect}_k$ *which can be completed to the VN-core* $(\mathrm{End}^\vee U) \oplus k$.

Proof. The von Neumann axiom

$$\mu_3(1 \otimes S \otimes 1)\delta_3 = 1$$

becomes the diagram (in which we have omitted "⊗"):

$$
\begin{tikzcd}
 & U(A)^* \; UA \; U(A)^* \; UA \; U(A)^* \; UA \arrow[rr, "1\;1\;S\;1\;1"] \arrow[dd, "1\;1\;1\;d\;1\;1"] & & U(A)^* \; UA \; U(A^*)^* \; U(A^*) \; U(A)^* \; UA \arrow[dr] & \\
 & & U(A)^* \; UA \; U(A^*) \; U(A^*)^* \; U(A)^* \; UA \arrow[ur, "1\;1\;c\;1\;1"] \arrow[rr, "1\;c\;1"'] & & U(A)^* \; U(A^*)^* \; U(A)^* \; UA \; U(A^*) \; UA \\
 & U(A)^* \; UA \; U(A)^* \; U(A)^{**} \; U(A)^* \; UA \arrow[ur, "1\;1\;u^{-1}\;u^*\;1\;1"] \arrow[rrr, "1\;c\;1"'] \arrow[d, phantom, "(*)"] & & & U(A)^* \; U(A)^{**} \; U(A)^* \; UA \; U(A)^* \; UA \arrow[u, "1\;u^*\;1\;1\;u^{-1}\;1"'] \arrow[d, "\tilde{\rho}^*\;\tilde{\tau}"] \\
 & U(A)^* \; U(A) \; U(A)^* \; UA \arrow[uuu, bend left, "1\;1\;1\;n\;1"] \arrow[urrr, "e^*\;1\;1\;1"] \arrow[r, "U(e)^*\;1"'] \arrow[drr, "1\;\tilde{\tau}"] \arrow[ddr, "1\;e"] \arrow[dd, phantom, "(\mathrm{E1})"] & U(A\;A^*\;A)^* \; U(A) \; U(A)^* \; UA \arrow[rr, "1\;\tilde{\tau}"] & & U(A\;A^*\;A)^* \; U(A\;A^*\;A) \arrow[dd, "\mathrm{cop}_{C=A\;A^*\;A}"] \\
 & & & U(A)^* \; U(A\;A^*\;A) \arrow[ur, "U(e)^*\;1"'] \arrow[dl, "1\;Ue"] & \\
U(A)^* \; U(A) \arrow[uuuuur, bend left, "\delta_3"] \arrow[uur, "1\;n\;1"] \arrow[rr, "1"] & & U(A)^* \; UA \arrow[rr, "\mathrm{cop}_{C=A}"] & & \int^C U(C)^* \; U(C)
\end{tikzcd}
$$

where $(*)$ is the exterior of the diagram

$$
\begin{tikzcd}
U(A)^* \otimes UA \otimes U(A)^* \otimes UA \otimes U(A)^* \otimes UA \arrow[rr, "1\otimes 1\otimes 1\otimes d\otimes 1\otimes 1"] \arrow[dr, "1\otimes c\otimes 1"] & & U(A)^* \otimes UA \otimes U(A)^* \otimes U(A)^{**} \otimes U(A)^* \otimes UA \arrow[dd, "1\otimes c\otimes 1"] \\
 & U(A)^* \otimes UA \otimes U(A)^* \otimes U(A) \otimes U(A)^* \arrow[dr, "1\otimes d\otimes 1\otimes 1\otimes 1"] \arrow[d, phantom, "(\mathrm{E2})"] & \\
U(A)^* \otimes UA \otimes U(A)^* \otimes U(A) \arrow[uu, "1\otimes 1\otimes 1\otimes n\otimes 1"] \arrow[ur, "1\otimes n\otimes 1\otimes 1\otimes 1"] \arrow[rr, "e^*\otimes 1\otimes 1\otimes 1"'] & & U(A)^* \otimes U(A)^{**} \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes UA \\
U(A)^* \otimes UA \arrow[u, "1\otimes n\otimes 1"] & &
\end{tikzcd}
$$

which commutes using (E2) and commutativity of

$$
\begin{tikzcd}
 & \cdot \arrow[r, "1\;1\;n"] & \cdot \arrow[dd, "c"] \\
\cdot \arrow[ur, "n"] \arrow[dr, "n"'] & & \\
 & \cdot \arrow[r, "n\;1\;1"'] & \cdot
\end{tikzcd}
$$

□

3 Examples

3.1 Example

The first type of example is derived from the idea of a (contravariant) involution on a (small) comonoidal category $\mathcal{D}$. This includes the doubles $\mathcal{D} = \mathcal{B}^{\text{op}} + \mathcal{B}$ and $\mathcal{D} = \mathcal{B}^{\text{op}} \otimes \mathcal{B}$ with their respective "switch" maps (where $\mathcal{B}$ is a given small comonoidal $\mathbf{Vect}_k$-category), or any small comonoidal and compact-monoidal $\mathbf{Vect}_k$-category $\mathcal{D}$ (such as the category $\mathbf{Mat}_k$ of finite matrices over k) with the tensor duals of objects now providing an antipode on the comonoidal aspect of the structure rather than on the monoidal part, or any $*$-algebra structure on a given k-bialgebra (e.g., a C^*-bialgebra) with the $*$-operation providing the antipode.

In each case, an *even* functor from $\mathcal{D}$ to $\mathbf{Vect}$ is defined to be a (k-linear) functor F equipped with a (chosen) dinatural isomorphism

$$F(D^*) \cong F(D).$$

If we take the morphisms of even functors to be all the natural transformations between them then we obtain a category

$$\mathcal{E} = \mathcal{E}(\mathcal{D}, \mathbf{Vect}).$$

Let $\mathcal{A} = \mathcal{E}(\mathcal{D}, \mathbf{Vect}_{\text{fd}})_{\text{fs}}$ be the full subcategory of $\mathcal{E}$ consisting of the finitely valued functors of finite support. While this category is generally not compact, it has on it a natural antipode derived from those on $\mathcal{D}$ and $\mathbf{Vect}_{\text{fd}}$, namely

$$A^*(D) := A(D^*)^*.$$

Of course, there are also examples where $\mathcal{A}$ is actually compact, such as those where $\mathcal{D}$ is a Hopf algebroid, in the sense of [4], with antipode $(-)^* = S$, in which case each A from $\mathcal{D}$ to $\mathbf{Vect}$ has a symmetry structure on it.

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{E}$ consisting of the small coproducts in $\mathcal{E}$ of objects from $\mathcal{A}$. This category $\mathcal{C}$ is easily seen to be monoidal under the pointwise convolution structure from $\mathcal{D}$, and the inclusion $\mathcal{A} \subset \mathcal{C}$ is U-dense for the functor

$$U : \mathcal{C} \to \mathbf{Vect}_k$$

given by

$$U(C) = \sum_D C(D)$$

which is split semigroupal with UA finite dimensional for all $A \in \mathcal{A}$. Moreover,

$$\begin{aligned} U(A^*) &= \bigoplus_D A^*(D) \\ &= \bigoplus_D A(D)^* \\ &= U(A)^*, \end{aligned}$$

for all $A \in \mathcal{A}$. The conditions of (5) are easily verified if we define maps

$$e : A \otimes A^* \otimes A \to A$$

by commutativity of the diagrams

$$\begin{tikzcd}
A(D) \otimes A^*(D) \otimes A(D) \arrow[r, "e_D"] \arrow[d, "\cong"'] & A(D) \\
A(D) \otimes A(D)^* \otimes A(D), \arrow[ur, "1\otimes \mathrm{ev}"'] &
\end{tikzcd}$$

where the exterior of

$$\begin{tikzcd}
& A^*(D) \otimes A(D) \arrow[dl, "A^*(f)\otimes 1"'] \arrow[d, "\cong"'] \arrow[ddr, "\hat{e}"] & \\
& A(D)^* \otimes A(D) \arrow[dr, "\mathrm{ev}"] & \\
A^*(E) \otimes A(D) \arrow[r, "\cong"] \arrow[ddr, "1\otimes A(f)"'] & A(E)^* \otimes A(D) \arrow[u, "A(f)^*\otimes 1"] \arrow[d, "1\otimes A(f)"'] & k \\
& A(E)^* \otimes A(E) \arrow[ur, "\mathrm{ev}"] \arrow[d, "\cong"'] & \\
& A^*(E) \otimes A(E) \arrow[uur, "\hat{e}"'] &
\end{tikzcd}$$

commutes for all maps $f : D \to E$ in $\mathcal{D}$ so that

$$e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes k \cong A$$

is a genuine map in $\mathcal{C}$ when $\mathcal{C}$ is given the pointwise monoidal structure from $\mathcal{D}$. This completes the details of the general example.

3.2 Example

In the case where $k = \mathbb{C}$ and $\mathcal{D}$ has just one object D whose endomorphism algebra is a C^*-bialgebra, we have a one-object comonoidal category $\mathcal{D}$ with a $\mathbb{C}$-conjugate-linear antipode given by the $*$-operation. Then the convolution $[\mathcal{D}, \mathbf{Hilb}_{\mathrm{fd}}]$, where

$$[\mathcal{D}, \mathbf{Hilb}_{\mathrm{fd}}] \subset [\mathcal{D}, \mathbf{Vect}_{\mathbb{C}}],$$

is a monoidal category, with a $\mathbb{C}$-linear antipode given by

$$F^*(D) = F(D^*)^\circ$$

where H° denotes the conjugate-transpose of $H \in \mathbf{Hilb}_{\mathrm{fd}}$. We now interpret an even functor F to be a functor equipped with a dinatural isomorphism $F(D^*) \cong F(D)$ in $D \in \mathcal{D}$ which is $\mathbb{C}$-*linear*, so that $F^*(D) \cong F(D)^\circ$ for such a functor.

Take $\mathcal{A} = \mathcal{E}(\mathcal{D}, \mathbf{Hilb}_{\mathrm{fd}})$ and let $\mathcal{C}$ be the class of small coproducts in $[\mathcal{D}, \mathbf{Vect}_{\mathbb{C}}]$ of the underlying $[\mathcal{D}, \mathbf{Vect}_{\mathbb{C}}]$-representations of A's in $\mathcal{A}$ (with the appropriate maps). Each map

$$e : A \otimes A^* \otimes A \to A$$

in $\mathcal{C}$ is defined by the $\mathbb{C}$-linear components

$$e : A(D) \otimes A^*(D) \otimes A(D) \xrightarrow{1\otimes\hat{e}} A(D),$$

where

$$\hat{e} : A^*(D) \otimes A(D) \to \mathbb{C}$$

in $\mathbf{Vect}_{\mathbb{C}}$ comes from the $\mathbb{C}$-bilinear composite of two maps which are both $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-linear in the second, namely

$$\begin{tikzcd} A^*(D) \times A(D) \arrow[r] \arrow[d, "\cong"'] & \mathbb{C} \\ A(D)^\circ \times A(D). \arrow[ur, "\langle -,- \rangle"'] & \end{tikzcd}$$

The remainder of this example is as seen before in Example 3.1.

3.3 Example

Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a (small) braided monoidal category and let $\mathcal{B}$ be the k-linearization of $\mathbf{Semicoalg}(\mathcal{V})$ with the monoidal structure induced from that on $\mathcal{V}$. By analogy with [5], let $\mathcal{X} \subset \mathcal{B}$ be a finite full subcategory of $\mathcal{B}$ with $I \in \mathcal{X}$ and $\mathcal{X}^{\mathrm{op}}$ promonoidal when

$$p(x, y, z) = \mathcal{B}(z, x \otimes y)$$
$$j(z) = \mathcal{B}(z, I)$$

for $x, y, z \in \mathcal{X}$.

For example (cf. [5]), one could take $\mathcal{X}$ to be a (finite) set of non-isomorphic "basic" objects in some braided monoidal category $\mathcal{V}$, where each $x \in \mathcal{X}$ has a coassociative diagonal map $\delta : x \to x \otimes x$. However, we won't need the category $\mathcal{X}$ to be discrete or locally finite in the following.

Now let $\mathcal{C}$ be the convolution $[\mathcal{X}^{\mathrm{op}}, \mathbf{Vect}]$ and let $\mathcal{A} = [\mathcal{X}^{\mathrm{op}}, \mathbf{Vect}_{\mathrm{fd}}]$. The functor

$$U : \mathcal{C} \to \mathbf{Vect}$$

is defined by

$$U(C) = \bigoplus_x C(x),$$

and the obvious inclusion $\mathcal{A} \subset \mathcal{C}$ is U-dense. If there is a canonical (natural) retraction

$$\begin{tikzcd} p(x,y,z) = \mathcal{B}(z, x \otimes y) \arrow[r, shift right, "i_{x,y}"'] & \mathcal{B}(z,x) \otimes \mathcal{B}(z,y), \arrow[l, shift right, "r_{x,y}"'] \end{tikzcd}$$

derived from the semicoalgebra structures on x, y, z, then U becomes a split semigroupal functor via the structure maps

$$\begin{tikzcd} U(C) \otimes U(D) \arrow[r, shift left, "r"] \arrow[d, equal] & U(C \otimes D) \arrow[l, shift left, "i"] \arrow[d, equal] \\ \bigoplus_x C(x) \otimes \bigoplus_y D(y) \arrow[d, shift left, "\Delta^*"] & \bigoplus_z \int^{xy} p(x,y,z) \otimes C(x) \otimes D(y) \arrow[d, shift left, "``i"'"] \\ \bigoplus_z C(z) \otimes D(z) \arrow[u, shift left, "\Delta"] & \bigoplus_z \int^{xy} \mathcal{B}(z,x) \otimes \mathcal{B}(z,y) \otimes C(x) \otimes D(y), \arrow[l, "\cong"'] \arrow[u, shift left, "``r"'"] \end{tikzcd}$$

where the isomorphism follows from the Yoneda lemma, and $ri = 1$.

If $\mathcal{X}$ also has on it a duality

$$(-)^* : \mathcal{X} \to \mathcal{X}^{\mathrm{op}}$$

such that $x \cong x^{**}$, then, on defining

$$A^*(x) = A(x^*)^*,$$

we obtain

$$\begin{aligned} U(A^*) &= \bigoplus_x A^*(x) \\ &= \bigoplus_x A(x^*)^* \\ &\cong \bigoplus_x A(x)^* && \text{since } x \cong x^{**} \\ &\cong U(A)^*, \end{aligned}$$

for $A \in \mathcal{A}$, in accordance with the fourth requirement on U.

Finally, to obtain a suitable map

$$e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes I \cong A,$$

where $\hat{e} : A^* \otimes A \to I$, we suppose each A in $\mathcal{A}$ has on it a "dual coupling"

$$\chi = \chi_{xy} : A(x)^* \otimes A(y) \to \mathcal{B}(x^* \otimes y, I).$$

By considering the Yoneda expansion

$$A(x) \cong \int^z A(z) \otimes \mathcal{X}(x, z)$$

of the various functors A in $\mathcal{A} = [\mathcal{X}^{\mathrm{op}}, \mathbf{Vect}_{\mathrm{fd}}]$, such a coupling exists on each A if we suppose merely that $\mathcal{X}$ itself is "coupled" by a natural transformation

$$\chi : \mathcal{X}(y, z) \to \mathcal{X}(x, z) \otimes \mathcal{B}(x^* \otimes y, I);$$

or simply

$$\chi : \mathcal{X}(x, z)^* \otimes \mathcal{X}(y, z) \to \mathcal{B}(x^* \otimes y, I),$$

if $\mathcal{X}$ is locally finite. Then, the composite natural transformation

$$\begin{array}{c} A(x^*)^* \otimes A(y) \otimes \mathcal{B}(z, x \otimes y) \\ \big\downarrow{\scriptstyle \chi \otimes 1} \\ \mathcal{B}(x^{**} \otimes y, I) \otimes \mathcal{B}(z, x \otimes y) \\ \big\downarrow{\scriptstyle \cong} \\ \mathcal{B}(x \otimes y, I) \otimes \mathcal{B}(z, x \otimes y) \\ \big\downarrow{\scriptstyle \text{comp'n}} \\ \mathcal{B}(z, I) \end{array}$$

yields the map

$$\begin{array}{ccc} A^* \otimes A & \xrightarrow{\hat{e}} & I \\ \| & & \| \\ \int^{xy} A^*(x) \otimes A(y) \otimes p(x,y,-) & \longrightarrow & \mathcal{B}(-,I) \end{array}$$

because $p(x,y,-) = \mathcal{B}(-, x \otimes y)$ (by definition). Thus suitable conditions on the coupling χ give (5).

Remark. Actually, this last example in which the basic promonoidal structure occurs as a canonical retract of a comonoidal structure is typical of many other examples which can be treated along similar lines.

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